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NEIGHBORHOOD AND RADII PROPERTIES FOR A CERTAIN CLASS OF UNIVALENT FUNCTION BASED OB q-ANALOGUE OF RUSEHEWEYH OPERATOR

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Abstract

By considering the q analogue of Ruseheweyh operator, a new subclass of univalent functions with negative coefficients is defined. Coefficient bounds, extreme points, distortion bounds, Radii of starlikeness, convexity and close to convexity of this class are obtained. The integral representation of q analogue of Ruseheweyh operator and neighborhood concept are also investigated.

1. Introduction

Let denote the class of all functions of the following form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in: |z| < 1\}$. Also $\mathcal{T} \subseteq$ consisting the functions of the type

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$$f(z) = z - \sum_{k=2}^{\infty} z_k z^k, \qquad (a_k \geqslant 0). \tag{2}$$

For $n \in$, $0 < q < 1$, we define

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q} \\ [n]_q! &= \begin{cases} [n]_q[n-1]_q \dots [1]_q & , \quad n = 1, 2, \dots \\ 1 & , \quad n = 0. \end{cases} \end{aligned} \tag{3}$$

A q analogue means that $[n]_q \to n$ as $q \to 1$.

The q analogue of Ruseheweyh operator defined by

$$\mathcal{R}_q^\lambda f(z) = z - \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{4}$$

where, $f(z) \in \mathcal{T}$, $[\alpha]_q$ and $[\alpha]_q!$ are defined in (3), see [1].

From (4), we conclude that, if $q \to 1$, then

$$\lim_{q\to 1} \mathcal{R}_q^\lambda f(z) = z - \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{(\lambda)!(k-1)!} a_k z^k = \mathcal{R}^\lambda f(z), \tag{5}$$

where $\mathcal{R}^\lambda$ is the Ruseheweyh differential operator, which was defined in [4].

The Ruseheweyh differential operator $\mathcal{R}^\lambda$ has been studied by many authors, for example see [3, 6] and [7].

A function $f(z)$ belonging to the class $\mathcal{T}$ is in the class $\mathcal{R}_q^\lambda(\alpha, \beta)$ if it satisfies the condition:

$$Re\left\{ \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'} \right\} > \alpha \left| \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'} - 1 \right| + \beta, \tag{6}$$

where, $\alpha \geqslant 0$, $0 \leqslant \beta < 1$ and $\mathcal{R}_q^\lambda f(z)$ is defined by (4).

2. Coefficient Bounds and Extreme Points

In this section we obtain coefficient estimate and extreme points for functions in $\mathcal{R}_q^\lambda(\alpha, \beta)$.

Theorem 2.1 : Let $f(z) \in \mathcal{T}$. Then $f(z)$ is in $\mathcal{R}_q^\lambda(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha) - k(\alpha+\beta)][k+\lambda-1]_q!}{(1-\beta)[\lambda]_q![k-1]_q!} \alpha_k < 1. \tag{7}$$

Proof : Suppose that $f(z) \in \mathcal{T}$. By using the fact that

$$ReW > \alpha|W-1| + \beta \Longleftrightarrow Re\{W(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}\} > \beta,$$

where $\alpha \geqslant 0$, $0 \leqslant \beta < 1$, $\gamma \in \mathbb{R}$ and W by any complex number, and letting $W = \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}$ in (6) we obtain

$$Re\Big\{\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}\Big\} > \beta$$

or

$$Re\left\{\frac{\frac{z-\sum_{k=2}^{\infty}[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k}{z\Big(1-\sum_{k=2}^{\infty}k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}\Big)}(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma} - \beta\right\} > 0.$$

Then

$$Re\left\{\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k - \alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}}\right\} > 0.$$

The above inequality must hold for all z in U. Letting $z = re^{-i\theta}$, where $0 \leqslant r < 1$, we conclude:

$$Re\left\{\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k r^k - \alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k r^{k-1}}{1-\sum_{k=2}^{\infty}k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k r^{k-1}}\right\} > 0.$$

By letting $r \to 1$, through half line $z = re^{-i\theta}$ $(0 \leqslant r < 1)$ and by the mean value theorem, we get:

$$Re\left\{(1-\beta) - \sum_{k=2}^{\infty}\left[(1-\beta k) - \alpha(1-k)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right]a_k\right\} > 0.$$

Therefore

$$\sum_{k=2}^{\infty}\Big((1-\beta k) + \alpha(1-k)\Big)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k < 1-\beta,$$

and then

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha) - k(\alpha+\beta)][k+\lambda-1]_q!}{(1-\beta)[\lambda]_q![k-1]_q!}\alpha_k < 1.$$

Conversely, let (2) hold, we will show that (6) is satisfied and so $f(z) \in \mathcal{R}_q^\lambda(\alpha,\beta)$. By the fact that

$$ReW > \alpha \Longleftrightarrow |W-(1+\alpha)| < |W+(1-\alpha)|,$$

it is enough to show that:

$$\begin{aligned} L &= \left| \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'} - \Big(1+\alpha\left|\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}-1\right|+\beta\Big)\right| \\ &< \left| \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'} + \Big(1-\alpha\left|\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}-1\right|-\beta\Big)\right| = H. \end{aligned}$$

By letting $\eta = \frac{z(\mathcal{R}_q^\lambda f(z))'}{|z(\mathcal{R}_q^\lambda f(z))'|}$, we may write

$$\begin{aligned} H &= \left| \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'} + 1 - \beta - \alpha\left|\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}-1\right|\right| \\ &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left|\mathcal{R}_q^\lambda f(z) + (1-\beta)z(\mathcal{R}_q^\lambda f(z))' - \alpha\eta|\mathcal{R}_q^\lambda f(z) - z(\mathcal{R}_q^\lambda f(z))'|\right| \\ &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left| z - \sum_{k=2}^\infty \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k + (1-\beta)\Big(z - \sum_{k=2}^\infty k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\Big)\right. \\ &\left. -\alpha\eta\left| z - \sum_{k=2}^\infty \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} - z + \sum_{k=2}^\infty k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\right|\right| \\ &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left| 2x - \beta z - \sum_{k=2}^\infty \Big[1+(1-\beta)k+\alpha-\alpha k\Big]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\right|. \end{aligned}$$

By letting $z = re^{-i\theta}$ and $r \to 1$, we have:

$$H > \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left\{(2-\beta) - \sum_{k=2}^\infty \Big[k + (1-\beta\alpha) - k(\alpha+\beta)\Big]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k\right\},$$

and

$$\begin{aligned} L &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left|\mathcal{R}_q^\lambda f(z) - (1+\beta)z(\mathcal{R}_q^\lambda f(z))' - \alpha\eta\left|\mathcal{R}_q^\lambda f(z) - z(\mathcal{R}_q^\lambda f(z))'\right|\right| \\ &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left| z - \sum_{k=2}^\infty \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k - (1+\beta)\Big(z - \sum_{k=2}^\infty k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\Big)\right. \\ &\left. -\ \alpha\eta\left|\sum_{k=2}^\infty k\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k - \sum_{k=2}^\infty \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\right|\right| \\ &= \frac{1}{|z(\mathcal{R}_q^\lambda f(z))'|}\left| -\beta z - \sum_{k=2}^\infty \Big[1-(1+\beta)k-\alpha\Big]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k\right|. \end{aligned}$$

By letting $z = re^{-i\theta}$, $r \to 1$ and $|\eta| = 1$, we get:

$$L < \frac{|z|}{|z(\mathcal{R}_q^\lambda f(z))'|}\left[\beta + \sum_{k=2}^{\infty}\Big[-k + (1+\alpha) - k(\alpha+\beta)\Big]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k\right].$$

It is easy to verify that $H - L > 0$ if (2) holds, and so that proof is complete. □

Theorem 2.2 : Let $f_1(z) = z$ and $f_k(z) = z - \frac{(1-\beta)[\lambda]_q![k-1]_q!}{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}z^k$, where $k = 2, 3, \ldots$. Then $f \in \mathcal{R}_q^\lambda(\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum\limits_{k=1}^{\infty} t_k f_k(z)$, where $t_k \geqslant 0$ and $\sum\limits_{k=1}^{\infty} t_k = 1$. In particular, the extreme point of $\mathcal{R}_q^\lambda(\alpha,\beta)$ are the functions $f(z) = z$ and

$$f_k(z) = z - \frac{(1-\beta)[\lambda]_q![k-1]_q!}{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}z^k, \qquad (k = 2, 3, \ldots).$$

Proof : Let f be expressed in the form $f(z) = \sum\limits_{k=1}^{\infty} t_k f_k(z)$.This means that we can write

$$\begin{aligned} f(z) &= \sum_{k=2}^{\infty} t_k f_k(z) = t_1 f_1(z) + \sum_{k=1}^{\infty} t_k f_k(z) \\ &= t_1 z + \sum_{k=2}^{\infty} t_k z - \sum_{k=2}^{\infty}\frac{(1-\beta)[\lambda]_q![k-1]_q!}{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}t_k z^k \\ &= z\sum_{k=1}^{\infty} - \sum_{k=2}^{\infty} d_k z^k \\ &= z - \sum_{k=2}^{\infty} d_k z^k, \end{aligned}$$

where

$$d_k = \frac{(1-\beta)[\lambda]_q![k-1]_q!}{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}t_k.$$

Therefore $f \in \mathcal{R}_q^\lambda(\alpha,\beta)$. Since

$$\frac{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}{(1-\beta)[\lambda]_q![k-1]_q!}d_k = \sum_{k=1}^{\infty} d_k = 1 - d_1 < 1.$$

Conversely, suppose that $f \in \mathcal{R}_q^\lambda(\alpha,\beta)$. Then by setting $t_1 = 1 - \sum\limits_{k=1}^{\infty} t_k$, where

$$t_k = \frac{(1-\beta)[\lambda]_q![k-1]_q!}{[(1+\alpha)-k(\alpha+\beta)][k+\lambda-1]_q!}a_k, \qquad (k = 2, 3, \ldots),$$

we conclude that required result. □

3. Distortion bounds and Radii Properties

In this section, we obtain the distortion bounds for $\mathcal{R}_q^\lambda f(z)$, $\mathcal{R}^\lambda f(z)$ and f(z).

Theorem 3.1 : Let $f(z) \in \mathcal{R}_q^\lambda(\alpha, \beta)$, then

$$|z| - \frac{1-\beta}{1-\beta-(\alpha+\beta)}|z|^2 < |\mathcal{R}_q^\lambda f(z)| < |z|\frac{1-\beta}{1-\beta-(\alpha+\beta)}|z|^2. \tag{8}$$

Proof : For $f \in \mathcal{R}_q^\lambda(\alpha, \beta)$, by (2) we have

$$\sum_{k=1}^{\infty} a_k \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} < \frac{1-\beta}{(1-\beta)-(\alpha+\beta)}.$$

Therefore

$$\begin{aligned}
|\mathcal{R}_q^\lambda f(z)| &= \left| z - \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k \right| \\
&\leqslant |z| + |z|^2 \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k \\
&< |z| \frac{1-\beta}{1-\beta-(\alpha+\beta)}|z|^2,
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{R}_q^\lambda f(z)| &\geqslant |z| - |z|^2 \sum_{k=2}^{\infty} a_k \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} \\
&> |z| - \frac{1-\beta}{1-\beta-(\alpha+\beta)}|z|^2.
\end{aligned}$$

□

Corollary 3.1 : If $q \to 1$, we conclude the distortion bounds for the Ruseheweyh differential operator $\mathcal{R}^\lambda f(z)$, and when $\lambda = 0$, we get the distortion bounds for $f(z)$.

Theorem 3.3 : Let $f \in \mathcal{R}_q^\lambda(\alpha, \beta)$. Then $f(z)$ is starlike of order γ $(0 \leqslant \gamma < 1)$ convex of order θ $(0 \leqslant \theta < 1)$ and close-to-convex of order τ $(0 \leqslant \tau < 1)$ in $|z| < R_1(\gamma, \alpha, \beta, \lambda, q)$,

$|z| < R_2(\theta, \alpha, \beta, \lambda, q)$ and $|z| < R_3(\tau, \alpha, \beta, \lambda, q)$ respectively, where

$$R_1(\gamma, \alpha, \beta, \lambda, q) = \inf_k \left[\frac{((1+\alpha) - k(\alpha+\beta))(1-\gamma)[k+\lambda-1]_q!}{(1-\beta)[\lambda]_q![k-1]_q!(k-\gamma)} \right]^{\frac{1}{k-1}}, \tag{9}$$

$$R_2(\theta, \alpha, \beta, \lambda, q) = \inf_k \left[\frac{((1+\alpha) - k(\alpha+\beta))(1-\theta)[k+\lambda-1]_q!}{k(1-\beta)[\lambda]_q![k-1]_q!(k-\theta)} \right]^{\frac{1}{k-1}}, \tag{10}$$

$$R_3(\tau, \alpha, \beta, \lambda, q) = \inf_k \left[\frac{((1+\alpha) - k(\alpha+\beta))(1-\tau)[k+\lambda-1]_q!}{k(1-\beta)[\lambda]_q![k-1]_q!(k-\tau)} \right]^{\frac{1}{k-1}}. \tag{11}$$

Proof : For $0 \leqslant \gamma < 1$ we need to show that $\left|\frac{zf'}{f} - 1\right| < 1 - \gamma$. In other words, it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{z - \sum\limits_{k=2}^{\infty} k a_k z^{k-1}}{z - \sum\limits_{k=2}^{\infty} a_k z^k} - 1\right| = \left|\frac{\sum\limits_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} a_k z^{k-1}}\right|$$

$$< \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} a_k |z|^{k-1}} < 1 - \gamma,$$

or

$$\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1} < 1 - \gamma - (1-\gamma)\sum_{k=2}^{\infty} a_k|z|^{k-1},$$

or

$$\sum_{k=2}^{\infty} \frac{k-\gamma}{1-\gamma} a_k |z|^{k-1} < 1.$$

By (7) it is easy to see that above inequality holds if

$$|z|^{k-1} < \frac{[(1+\alpha) - k(\alpha+\beta)](1-\gamma)[k+\lambda-1]_q!}{(1-\beta)(k-\gamma)[\lambda]_q![k-1]_q!},$$

and this gives (9).

Since f is convex if and only if zf' is starlike, we obtain (10).

For the last relation (11), we most show that $|f(z) - 1| \leqslant 1 - \tau$, for $|z| < R_3(\tau, \alpha, \beta, \lambda, q)$.

But

$$|f'(z)-1| = \left|\sum_{k=2}^{\infty} ka_k z^{k-1}\right|$$
$$\leqslant \sum_{k=2}^{\infty} ka_k|z|^{k-1}.$$

Thus, $|f'(z)-1| < 1-\tau$ if $\sum\limits_{k=2}^{\infty} \frac{ka_k}{\tau-1}|z^{k-1}| \leqslant 1$. But by Theorem 3.1, above inequality holds true if

$$|z|^{k-1} < \frac{[(1+\alpha)-k(\alpha+\beta)](1-\tau)[k+\lambda-1]_q!}{k(1-\beta)[\lambda]_q![k-1]_q!},$$

This gives the relation (11), so that proof is complete. □

4. Integral Representation and Neighborhood Property

In the last section we obtain integral representation for $\mathcal{R}_q^\lambda f(z)$ and investigate about the neighborhood concept for the class $\mathcal{R}_q^\lambda(\alpha,\beta)$.

Theorem 4.1 : Let $f(z) \in \mathcal{R}_q^\lambda(\alpha,\beta)$, then

$$\mathcal{R}_q^\lambda f(z) = \exp\Big(\int_0^z \frac{G(t)-\alpha}{(\beta G(t)-\alpha)^t}dt\Big), \qquad (|G(z)|<1, \quad z\in U).$$

Proof : For $f(z) \in \mathcal{R}_q^\lambda(\alpha,\beta)$ and $W = \frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}$ by (6), we have

$$Re\{W\} > \alpha|W-1| + \beta.$$

Thus $\left|\frac{W-1}{W-\beta}\right| < \frac{1}{\alpha}$ or $\frac{W-1}{W-\beta} = \frac{G(z)}{\alpha}$, where $|G(z)|<1$, $z \in U$. So we obtain

$$\frac{\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}-1}{\frac{\mathcal{R}_q^\lambda f(z)}{z(\mathcal{R}_q^\lambda f(z))'}-\beta} = \frac{G(z)}{\alpha},$$

or

$$\frac{\mathcal{R}_q^\lambda f(z) - z(\mathcal{R}_q^\lambda f(z))'}{\mathcal{R}_q^\lambda f(z) - \beta z(\mathcal{R}_q^\lambda f(z))'} = \frac{G(z)}{\alpha}.$$

Therfore

$$\frac{(\mathcal{R}_q^\lambda f(z))'}{\mathcal{R}_q^\lambda f(z)} = \frac{G(z)-\alpha}{(\beta G(z)-\alpha)z},$$

which is equivalent by integration

$$\log(\mathcal{R}_q^\lambda f(z)) = \int_0^z \frac{G(z)-\alpha}{(\beta G(t)-\alpha)^t}dt,$$

and this gives the required result. □

Now, we define the (k,δ)-neighborhood of a function $f(z) \in \mathcal{T}$ by

$$\mathcal{N}k,\delta(f) = \Big\{g \in \mathcal{T} : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|a_k - b_k| \leqslant \delta\Big\}. \tag{13}$$

For the identity function $I(z) = z$, we have:

$$\mathcal{N}k,\delta(I) = \Big\{g \in \mathcal{T} : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|b_k| \leqslant \delta\Big\}. \tag{15}$$

Also we say that $g(z) = z - \sum\limits_{k=2}^{\infty} b_k z^k \in \mathcal{R}_q^\lambda(\alpha,\beta,\theta)$ if there exists a function $f \in \mathcal{R}_q^\lambda(\alpha,\beta)$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| \leqslant 1-\theta, \qquad (z \in U, \quad 0 \leqslant \theta < 1). \tag{16}$$

The concept of neighborhood has been studied by many authors, for example see [2, 5] and [8].

Theorem 4.2 : Let

$$\delta = \frac{1-\beta}{1-\beta-(\alpha+\beta)}, \tag{17}$$

then, $\mathcal{R}_q^\lambda(\alpha,\beta) \subset \mathcal{N}k,\delta(I)$.

Proof : By corollary of Theorem 3.3, we have

$$g'(z) \leqslant \left[1 + \frac{1-\beta}{1-\beta-(\alpha+\beta)}|z|\right].$$

Indeed

$$|g'(z)| = \left|1 - \sum_{k=2}^{\infty} k b_k z^{k-1}\right| \leqslant 1 + \sum_{k=2}^{\infty} k b_k |z|^{k-1}.$$

By choosing the values of z on the real axis and then $z \to 1^-$, through real values we obtain

$$\sum_{k=2}^{\infty} k b_k \leqslant \frac{1-\beta}{1-\beta-(\alpha+\beta)} = \delta.$$

So, $g(z) \in \mathcal{N}k, \delta(I)$. □

Theorem 4.3 : If $f(z) \in \mathcal{R}_q^\lambda(\alpha, \beta)$ and

$$\theta = 1 - \frac{\delta(1-\alpha-2\beta)[1+\lambda]_q!}{2(1-\alpha-2\beta)[1+\lambda]_q! - 2(1-\beta)[\lambda]_q!} \tag{20}$$

then, $\mathcal{N}k, \delta(f) \subset \mathcal{R}_q^\lambda(\alpha, \beta, \theta)$.

Proof : Let $g \in \mathcal{N}k, \delta(f)$, then we have from (12), that

$$\sum_{k=2}^{\infty} k|a_k - b_k| < \delta,$$

which implies the coefficient inequality $\sum\limits_{k=2}^{\infty} |a_k - b_k| \leqslant \frac{\delta}{2}$. Also since $f \in \mathcal{R}_q^\lambda(\alpha, \beta)$, we have from (7),

$$\sum_{k=2}^{\infty} a_k \leqslant \frac{(1-\beta)[\lambda]_q![1]_q}{(1-\alpha-2\beta)[1+\lambda]_q!}.$$

So, we get

$$\begin{aligned} \left|\frac{g(z)}{f(z)} - 1\right| &< \left|\frac{\sum\limits_{k=2}^{\infty}(a_k - b_k)z^k}{z - \sum\limits_{k=2}^{\infty} a_k z^k}\right| < \frac{\sum\limits_{k=2}^{\infty}|a_k - b_k|}{1 - \sum\limits_{k=2}^{\infty} a_k z^k} \\ &\leqslant \frac{\delta}{2}\frac{(1-\alpha-2\beta)[1+\lambda]_q!}{(1-\alpha-2\beta)[1+\lambda]_q! - (1-\beta)[\lambda]_q!} \\ &= 1 - \theta \end{aligned} \tag{23}$$

Thus, by (17), $g \in \mathcal{R}_q^\lambda(\alpha, \beta, \theta)$, for θ, gives by (16). □

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